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Classification of Nilpotent Lie Superalgebras of Dimension Five. I

Ahmad Hegazi[1]

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In this paper a classification is made of all nilpotent Lie superalgebras (graded Lie algebras) of maximum dimension five.

A Lie superalgebra $L = L_0 \oplus L_1$ is a superalgebra over a base field $K = R$ or C with an operation [. , .] satisfying the following axioms:

(i) $[x_\alpha, x_\beta] = -(-1)^{\alpha,\beta} [x_\beta, x_\alpha]$

(ii) $(-1)^{\alpha,\gamma}[[x_\alpha, x_\beta], x_\gamma] + (-1)^{\alpha,\beta} [[x_\beta, x_\alpha], x_\gamma] + (-1)^{\gamma,\beta} [[x_\gamma, x_\alpha], x_\beta] = 0$, $x_\alpha \in L_\alpha$, $x_\beta \in L_\beta$; $\alpha, \beta \in \{0,1\} = Z_2$

L_0 is called the even part, and is a Lie algebra, and L_1 is called the odd part, and is an L_0-module by restriction of the adjoint representation.[(1)] We say that $L = L_0 \oplus L_1$ and $L' = L'_0 \oplus L'_1$ are equivalent if there are isomorphisms $L_0 \to L'_0$ and $L_1 \to L'_1$ which preserve the bracket multiplication. We say also that L is trivial if $[L_1, L_1] = \{0\}$; otherwise L is nontrivial. It is also worth noting that the structure constants of a trivial Lie superalgebra L can be interpreted as the structure constants of an associated Lie algebra L' provided that we replace the zero anticommutator of L by the zero commutator of L'. However, under this correspondence, inequivalent Lie superalgebras can lead to equivalent Lie algebras. Other departures from ordinary Lie theory include the fact that Lie's theorem is not valid; that Cartan's criterion for simplicity only works in one direction; that there is no obvious analog of Levi's theorem; and that there can exist zero divisors in the enveloping

[1] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt; e-mail: Sinfac@mum.mans.eun.eg.

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algebra.[1,5] In the present work we give a classification of nilpotent Lie superalgebras, which are not Lie algebras, up to dimension five—the classification of Lie superalgebras of dimension four does not encounter any simple Lie superalgebra (the smallest simple Lie superalgebras is of dimension five, the so-called di-spin algebra[5]—and all of these are solvable.[6–8]

Let $L = L_0 \oplus L_1$ be a Lie superalgebra; define a sequence of ideals of L by $L^{(0)} = L$, $L^{(1)} = [L, L]$, $L^{(2)} = [L, L^{(1)}]$, . . . , $L^{(i)} = [L, L^{(i-1)}]$. Then L is called nilpotent if there exists i such that $L^{(i)} = (0)$; i is called the degree of nilpotency. An ideal I of L is superideal if $\sigma(I) = I$, where σ is an automorphism of L defined by $\sigma(x_0 + x_1) = (x_0 - x_1)$ for $x_0 \in L_0$, $x_1 \in L_1$.

For a Lie superalgebra L, it is known that L is nilpotent if and only if ad_x is a nilpotent operator for all $x \in L$, where ad_x is the adjoint representation of L.[1,9] It is also known that the Engel theorem is valid but the Lie theorem is not valid for nilpotent Lie superalgebras.[1,5] Now we study the nilpotent Lie superalgebra over $K = R$ or C with dimension ≤ 5 and such that $[L, L] \neq 0$, otherwise all results are trivial.

We tabulate the families of equivalence classes of the indecomposable nilpotent Lie superalgebra of maximum dimension five (Table I). We say that $L = L_0 \oplus L_1$ is an (m, n) algebra if dim L_0 (resp. L_1) is m (resp. n). For the labeling of algebras, the letters A, B, C, D, E with integral superscript i denote equivalence classes of algebras of dimension d, $d = 1$ for A, $d = 2$ for B, $d = 3$ for C, $d = 4$ for D, and $d = 5$ for E. Here L is the associated Lie algebra of the trivial Lie superalgebra and the superscript i is omitted whenever its range is just the integer one.[6–8]

Proposition 1.1. For a Lie superalgebra of type $(1, n)$, either $[L_0, L_1] = (0)$ or $[L_1, L_1] = (0)$, and it is nilpotent if and only if there exists a basis $B_L = B_{L_0} \cup B_{L_1}$ such that either $[x_1, x_1] = a_i x_0$, $a_i \in \{-1,0,1]$ for $K = R$, $a_i \in \{0,1\}$ for $K = C$, $x_0 \in B_{L_0}$, $x_1 \in B_{L_1}$, and all other Lie products are zero, or the matrix representation of ad_{x_0} has the Jordan form

$$\begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 1 & 0 & & & & 0 \\ 0 & 0 & 0 & 0 & . & & & . \\ . & & . & . & 1 & . & & . \\ . & & & . & . & 0 & . & . \\ . & & & & . & . & 1 & 0 \\ 0 & & & & & 0 & 0 & 0 \\ 0 & 0 & . & . & . & 0 & 0 & 0 \\ & & & & & & & \end{bmatrix}$$

Table I

Type	L	Characterization	Relation	Comments	K
(1,0)	—	—	—	—	—
(0,1)	A	$L = \langle 0 \rangle \oplus \langle x_1 \rangle, [x_1, x_1] = 0$	Trivial	Equivalent to $A_{1,1}$ and Abelian	R, C
(1,1)	$(A_{1,1} + A)$	$L = \langle x_0 \rangle \oplus \langle x_1 \rangle, [x_1, x_1] = x_0$	$L^1_{(1,1)}$	Nontrivial	R, C
(2,0)	—	$L = \langle x_0, y_0 \rangle \oplus \langle 0 \rangle$	—	Abelian Lie algebra	R, C
(0,2)	—	—	—	—	—
(3,0)	—	$L = \langle x_0, y_0, z_0 \oplus \langle 0 \rangle, [x_0, y_0] = z_0$	$L^2_{(3,0)}$	Heisenberg Lie algebra	R, C
(2,1)	$(2A_{1,1} + A)$	$L = \langle x_0, y_0 \rangle \oplus \langle x_1 \rangle [x_1, x_1] = x_0, [x_0, y_0] = 0$	Derived from $L^1_{(1,1)}$	Nontrivial	R, C
(1,2)	C	$L = \langle x_0 \rangle \oplus \langle x_1, y_1 \rangle, [x_0, x_1] = y_1$	$L^3_{(1,2)}$	Trivial	R, C
(1,2)	$(A_{1,1} + 2A)^1$	$[x_1, x_1] = x_0, [y_1, y_1] = x_0$	$L^4_{(1,2)}$	Nontrivial	R, C
(1,2)	$(A_{1,1} + 2A)^2$	$[x_1, x_1] = x_0, [y_1, y_1] = x_0$	$L^5_{(1,2)}$	Nontrivial	R
(0,3)	—	—	—	—	—
(4,0)	—	$L = \langle x_0, y_0, z_0, v_0 \rangle, [x_0, y_0] = z_0, [x_0, z_0] = v_0$	$L^6_{(4,0)}$	Lie algebra	R, C
(3,1)	$(A_{3,1} + A)$	$L = \langle x_0, y_0, z_0 \rangle \oplus \langle x_1 \rangle, [x_0, y_0] = z_0, [x_1, x_1] = z_0$	$L^7_{(3,1)}$	Nontrivial Heisenberg Lie superalgebra	R, C
(2,2)	$(C + A)$	$L = \langle x_0, y_0 \rangle \oplus \langle x_1, y_1 \rangle, [x_0, x_1] = y_1, [x_1, x_1] = y_0$	$L^8_{(2,2)}$	Nontrivial	R, C
(2,2)	$(2A_{1,1} + 2A)^1$	$[x_1, x_1] = x_0, [y_1, y_1] = y_0, [x_1, y_1] = 1/2\ (x_0 + y_0)$	$L^9_{(2,2)}$	Nontrivial	R, C
(2,2)	$(2A_{1,1} + 2A)^2$	$[x_1, x_1] = x_0, [y_1, y_1] = y_0$	$L^{10}_{(2,2)}$	Nontrivial	R, C
(2,2)	$(2A_{1,1} + 2A)^3$	$[x_1, x_1] = x_0, [y_1, y_1] = y_0, [x_1, y_1] = (x_0 - y_0)$	$L^{11}_{(2,2)}$	Nontrivial	R, C
(2,2)	$(2A_{1,1} + 2A)^4$	$[x_1, x_1] = x_0, [y_1, y_1] = y_0, [x_1, y_1] = x_0$	$L^{12}_{(2,2)}$	Nontrivial	R, C
(1,3)	D	$L = \langle x_0 \rangle \oplus \langle x_1, y_1, z_1 \rangle, [x_0, x_1] = y_1, [x_0, y_1] = z_1$	$L^{13}_{(1,3)}$	Trivial	R, C
(1,3)	$(A_{1,1} + 3A)^1$	$[x_1, x_1] = x_0, [y_1, y_1] = x_0, [z_1, z_1] = x_0$	$L^{14}_{(1,3)}$	Nontrivial	R, C
(1,3)	$(A_{1,1} + 3A)^2$	$[x_1, x_1] = x_0, [y_1, y_1] = x_0, [z_1, z_1] = x_0$	$L^{15}_{(1,3)}$	Nontrivial	R, C
(0,4)	—	—	—	—	—
(5,0)	—	$L = \langle x_0, y_0, z_0, v_0, w_0 \rangle, [x_0, y_0] = w_0, [z_0, v_0] = w_0$	$L^{16}_{(5,0)}$	Heisenberg Lie algebra	R, C
(5,0)	—	$[x_0, y_0] = v_0, [x_0, z_0] = w_0$	$L^{17}_{(5,0)}$	Lie algebra	R, C
(5,0)	—	$[x_0, y_0] = z_0, [x_0, z_0] = v_0, [y_0, w_0] = v_0$	$L^{18}_{(5,0)}$	Lie algebra	R, C
(5,0)	—	$[x_0, y_0] = z_0, [x_0, z_0] = v_0, [y_0, z_0] = w_0$	$L^{19}_{(5,0)}$	Lie algebra	R, C

Table I—Continued

Type	L	Characterization	Relation	Comments	K
(5,0)	—	$[x_0, y_0] = z_0$, $[x_0, z_0] = v_0$, $[x_0, v_0] = w_0$ equivalent to $[x_0, y_0] = w_0$, $[x_0, z_0] = y_0$, $[x_0, v_0] = z_0$	$L^{20}_{(5,0)}$	Lie algebra	R, C
(5,0)	—	$[x_0, y_0] = z_0$, $[x_0, z_0] = v_0$, $[x_0, v_0] = w_0$, $[y_0, z_0] = w_0$	$L^{21}_{(5,0)}$	Lie algebra	R, C
(4,1)	E^1	$L = \langle x_0, y_0, z_0, v_0\rangle \oplus \langle x_1\rangle$, $[v_0, y_0] = x_0$, $[v_0, z_0] = z_0$, $[x_1, x_1] = x_0$	$L^{22}_{(4,1)}$	Nontrivial	R, C
(4,1)	E^2	$[x_0, y_0] = z_0$, $[v_0, y_0] = x_0$, $[x_1, x_1] = z_0$	$L^{23}_{(4,1)}$	Nontrivial	R, C
(3,2)	E^3	$L = \langle x_0, y_0, z_0\rangle \oplus \langle x_1, y_1\rangle$, $[x_0, y_0] = z_0$, $[x_0, y_1] = x_1$	$L^{24}_{(3,2)}$	Trivial	R, C
(3,2)	$(E^4)^1$	$[x_0, y_0] = z_0$, $[x_1, x_1] = z_0$, $[y_1, y_1] = z_0$	$L^{25}_{(3,2)}$	Nontrivial Heisenberg Lie superalgebra	R, C
(3,2)	$(E^4)^2$	$[x_0, y_0] = z_0$, $[x_1, x_1] = z_0$, $[y_1, y_1] = -z_0$	$L^{26}_{(3,2)}$	Nontrivial	R, C
(3,2)	(E^3)	$[x_0, y_0] = z_0$, $[x_0, y_1] = x_1$, $[y_1, y_1] = z_0$	$L^{27}_{(3,2)}$	Nontrivial	R, C
(2,3)	$(C + A)^1$	$L = \langle x_0, y_0\rangle \oplus \langle x_1, y_1, z_1\rangle$, $[x_0, y_1] = x_1$, $[y_1, y_1] = y_0$, $[z_1, z_1] = y_0$	$L^{28}_{(2,3)}$	Nontrivial	R, C
(2,3)	$(C + A)^2$	$[x_0, y_1] = x_1$, $[y_1, y_1] = y_0$, $[z_1, z_1] = -y_0$	$L^{29}_{(2,3)}$	Nontrivial	R, C
(2,3)	$(2A_{1,1} + 3A)^1$	$[x_1, x_1] = x_0$, $[y_1, y_1] = y_0$, $[z_1, z_1] = x_0 + y_0$	$L^{30}_{(2,3)}$	Nontrivial	R, C
(2,3)	$(2A_{1,1} + 3A)^2$	$[x_1, x_1] = x_0$, $[y_1, y_1] = y_0$, $[z_1, z_1] = -(x_0 + y_0)$	$L^{31}_{(2,3)}$	Nontrivial	R, C
(2,3)	$(2A_{1,1} + 3A)^3$	$[x_1, x_1] = x_0$, $[y_1, y_1] = y_0$, $[z_1, z_1] = x_0 - y_0$	$L^{32}_{(2,3)}$	Nontrivial	R, C
(2,3)	$(2A_{1,1} + 3A)^4$	$[x_1, x_1] = x_0$, $[y_1, y_1] = y_0$, $[x_1, z_1] = y_0$	$L^{33}_{(2,3)}$	Nontrivial	R, C
(2,3)	$(2A_{1,1} + 3A)^5$	$[x_1, x_1] = x_0$, $[y_1, y_1] = y_0$, $[x_1, z_1] = x_0 + y_0$	$L^{34}_{(2,3)}$	Nontrivial	R, C
(2,3)	$(2A_{1,1} + 3A)^6$	$[x_1, x_1] = x_0$, $[y_1, y_1] = y_0$, $[x_1, z_1] = x_0 - y_0$	$L^{35}_{(2,3)}$	Nontrivial	R, C
(2,3)	$(2A_{1,1} + 3A)^7$	$[x_1, x_1] = x_0$, $[x_1, z_1] = y_0$, $[y_1, z_1] = x_0$	$L^{36}_{(2,3)}$	Nontrivial	R, C
(2,3)	$(2A_{1,1} + 3A)^8$	$[x_1, y_1] = x_0$, $[y_1, z_1] = y_0$	$L^{37}_{(2,3)}$	Nontrivial	R, C
(1,4)	E^5	$L = \langle x_0\rangle \oplus \langle x_1, y_1, z_1, v_1\rangle$, $[x_0, y_1] = x_1$, $[x_0, v_1] = z_1$	$L^{38}_{(1,4)}$	Trivial	R, C
(1,4)	E^6	$[x_0, y_1] = x_1$, $[x_0, z_1] = y_1$, $[x_0, v_1] = z_1$	$L^{39}_{(1,4)}$	Trivial	R, C

Table I—Continued

Type	L	Characterization	Relation	Comments	K
(1,4)	$(A_{1,1} + 4A)^1$	$[x_1, x_1] = x_0, [y_1, y_1] = x_0, [z_1, z_1] = x_0, [v_1, v_1] = x_0$	$L^{40}_{(1,4)}$	Nontrivial	R, C
(1,4)	$(A_{1,1} + 4A)^2$	$[x_1, x_1] = x_0, [y_1, y_1] = x_0, [z_1, z_1] = x_0, [v_1, v_1] = -x_0$	$L^{41}_{(1,4)}$	Nontrivial	R
(1,4)	$(A_{1,1} + 4A)^3$	$[x_1, x_1] = x_0, [y_1, y_1] = x_0, [z_1, z_1] = -x_0, [v_1, v_1] = -x_0$	$L^{42}_{(1,4)}$	Nontrivial	R

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